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B-1

I Mid Sem. 2011-2012

Algebra I

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Instructions. All Questions carry equal marks.

1. (a) Let p be a prime number. Prove that any two groups of order p are isomorphic to each other.

   (b) Give two non-isomorphic groups of order 6.

2. Let p be a prime number and $\sqrt[p]{1} = \{z \mid z^{p^n} = 1, \text{ for some } n\}$ denote the group of p^nth roots of unity under multiplication. Prove that all its proper subgroups are cyclic.

3. Let G be a group and let $H \neq G$ be a subgroup of finite index. Prove that H contains a normal subgroup of G of finite index. (Hint: Look at the G action on G/H.)

4. State and prove the first isomorphism theorem for groups.

5. If $f : G \to H$ is an isomorphism of groups, then prove that f^{-1}, its (set-theoretic) inverse is a group isomorphism from H to G. Thereby, prove that the subset $Aut(G)$, consisting of group automorphisms of a group G is a subgroup of $Perm(G)$, the group of permutations of the set G.

6. Let $Z/7Z = \{0, 1, 2, 3, 4, 5, 6\}$ denote the group of integers mod 7, under the addition modulo 7. Determine which of the following permutations of this set are group automorphisms:

   (a) $(0, 1, 2, 3, 4, 5, 6)$

   (b) $(1, 2, 3, 4, 5, 6)$

   (c) $(1, 2, 4)(3, 6, 5)$

   (d) $(1, 3, 2, 6, 4, 5)$

   (e) $(1, 4, 2)(3, 5, 6)$